WNE Linear Algebra
Final Exam
Series A

29 January 2024

Please use separate sheets for different problems. Please use single sheet for all questions. Give reasons to your answers. Please provide the following data on each sheet

- **name, surname and your student number,**
- **number of your group,**
- **number of the corresponding problem and the series.**

Each problem is worth 10 marks. Each question is worth 4 marks.

Problems

Problem 1.
Let $V = \text{lin}((1, 1, 5, -2), (2, 3, 11, -5), (5, -2, 18, -3))$ be a subspace of $\mathbb{R}^4$.

a) find a basis $\mathcal{A}$ of the subspace V and the dimension of V,
b) find a system of linear equations which set of solutions is equal to V.

Problem 2.
Let $V \subset \mathbb{R}^4$ be a subspace given by the homogeneous system of linear equations

$$\begin{cases} x_1 + 2x_2 + 14x_3 + 13x_4 = 0 \\ x_1 + 3x_2 + 19x_3 + 17x_4 = 0 \\ x_1 + x_2 + 9x_3 + 9x_4 = 0 \end{cases}$$

a) find a basis and the dimension of the subspace V,
b) for which $t \in \mathbb{R}$ vector $v = (t, -1, 1, -1)$ belongs to V? For every such t find coordinates of v relative to basis $\mathcal{A}$.

Problem 3.
Let

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) which of the matrices are diagonalizable?
b) for each diagonalizable matrix A_i find a matrix $C_i \in M(3 \times 3; \mathbb{R})$ such that

$$C_i^{-1} A_i C_i = \begin{bmatrix} a_i & 0 & 0 \\ 0 & b_i & 0 \\ 0 & 0 & c_i \end{bmatrix},$$

where $a_i \geqslant b_i \geqslant c_i$.

Problem 4.
Let $\mathcal{A} = ((0, 1, 0), (1, 0, 0), (0, 0, 1))$ be an ordered basis of $\mathbb{R}^3$ and let $\mathcal{B} = ((1, 1), (1, 2))$ be an ordered basis of $\mathbb{R}^2$. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be linear transformation given by the matrix

$$M(\varphi)_{\mathcal{B}}^{\mathcal{A}} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix},$$

and let $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by the formula

$$\psi((x_1, x_2)) = (2x_1 + x_2, x_1).$$

a) find the matrix $M(\psi)_{\mathcal{B}}^{\mathcal{B}}$,
b) find the formula of $\varphi \circ \psi$.

Problem 5.
Let
$$V = \operatorname{lin}((1,0,1),(1,2,-1),(1,1,0)),$$
be a subspace of $\mathbb{R}^3$.

a) find an orthonormal basis of V,
b) find the orthogonal projection of $w = (0,0,3)$ onto $V^{\perp}$.

Problem 6.
Consider the following linear programming problem $-2x_2 - x_4 \to \min$ in the standard form with constraints
$$\begin{cases} 3x_1 + x_2 + 3x_3 - x_4 - 2x_5 = 24 \\ x_1 - x_4 - x_5 = 6 \end{cases} \quad \text{and } x_i \geqslant 0 \text{ for } i = 1,\dots,5.$$

a) which of the sets $\mathcal{B}_1 = \{2,4\}$, $\mathcal{B}_2 = \{2,3\}$, $\mathcal{B}_3 = \{1,5\}$ is basic feasible? Write the corresponding basic solution for all basic sets,
b) solve the linear programming problem using simplex method. Start from the basic feasible set taken from part a).

Questions

Question 1.
Let $V \subset \mathbb{R}^6$ be a subspace given by
$$V = \{(x_1,\dots,x_6) \in \mathbb{R}^6 \mid x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 0\}.$$
Let $S = M(S_V)_{st}^{st}$ be the matrix of the orthogonal reflection/symmetry about V. Let $v = (1,-2,3,-5,4,1) \in \mathbb{R}^6$. Does it follow that $S^{101}v \neq v$?

Solution 1.
Yes, it does. Since $v = P_V(v) + P_{V^{\perp}}(v)$ and $v \notin V$, we have $P_{V^{\perp}}(v) \neq 0$ and therefore
$$S^{101}v = Sv = P_V(v) - P_{V^{\perp}}(v) \neq P_V(v) + P_{V^{\perp}}(v).$$

Question 2.
Let $M \in M(2 \times 2;\mathbb{R})$ be a matrix such that $M = M^{\intercal}$. Assume that $v^{\intercal}Mv = 0$ for any $v \in \mathbb{R}^2$. Does it follow that $M = 0$?

Solution 2.
Yet, it does. Let
$$M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$
Then
$$\varepsilon_1^{\intercal}M\varepsilon_1 = a = 0, \quad \varepsilon_2^{\intercal}M\varepsilon_2 = c = 0,$$
and finally, for $v = (1,1)$
$$v^{\intercal}Mv = a + 2b + c = b = 0.$$

Question 3.
Let $A \in M(3 \times 3;\mathbb{R})$ be a matrix such that $\det A \neq 0$. Does it follow that
$$\det \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} > 0?$$
Hint: in the above matrix 0 denotes the 3-by-3 zero matrix.

Solution 3.
No, it does not.
$$\det \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} = (\det A)(\det(-A)) = (\det A)(-1)^3(\det A) = -(\det A)^2 < 0.$$

Question 4.
Let $A, B \in M(2 \times 2;\mathbb{R})$. Assume that $\det(A - \lambda B) = 0$ has two different solutions $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 \neq \lambda_2$ and matrix B is invertible. Does it follow that AB^{-1} is diagonalizable?

Solution 4.
Yes, it does.
$$\det(A-\lambda B)=\det\big((AB^{-1}-\lambda I)B\big)=(\det\big(AB^{-1}-\lambda I\big))(\det B)=0,$$
$$\Updownarrow$$
$$\det\big(AB^{-1}-\lambda I\big)=0.$$
Therefore 2-by-2 matrix AB^{-1} has two different eigenvalues and it is diagonalizable.

Question 5.
Let $L\subset\mathbb{R}^2$ be an affine line. Assume that $\pi_L((1,2))=(2,3)$, where π_L denotes the affine orthogonal projection onto L. Does it follow that
$$L=(3,2)+\operatorname{lin}((1,-1))?$$

Solution 5.
Yes, it does. Let $p=(1,2), q=(2,3)$. It follows that $(2,3)\in L$ and $\overrightarrow{pq}=(2,3)-(1,2)=(1,1)\in\overrightarrow{L}^{\perp}$. Therefore
$$L=(2,3)+\operatorname{lin}((1,-1))=((2,3)+(1,-1))+\operatorname{lin}((1,-1))=(3,2)+\operatorname{lin}((1,-1)).$$